

# The 24-Cell and Calabi-Yau Threefolds with Hodge Numbers (1,1)

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## Abstract

Calabi-Yau threefolds with  $h^{11}(X) = h^{21}(X) = 1$  are constructed as free quotients of a hypersurface in the ambient toric variety defined by the 24-cell. Their fundamental groups are  $SL(2, 3)$ ,  $\mathbb{Z}_3 \rtimes \mathbb{Z}_8$ , and  $\mathbb{Z}_3 \times Q_8$ .

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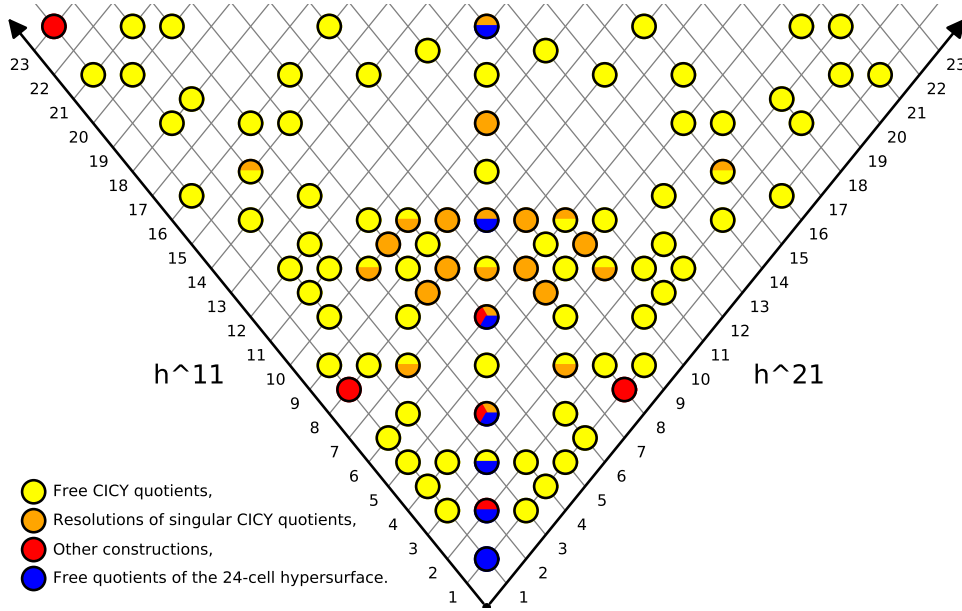
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**Figure 1:** Plot of the Hodge numbers of known Calabi-Yau threefolds (and their mirrors) with  $h^{11} + h^{21} < 25$ .

## 1 Introduction

The most basic observation about the Hodge numbers of Calabi-Yau threefolds is that they apparently cannot take arbitrary values, even though we do not have any good mathematical explanation. One empirical constraint [1] is that the height is limited to

$$h^{11} + h^{21} \leq 502, \quad (1)$$

leaving us only with a finite number of possible Hodge numbers. Not having made any progress in the way of an upper bound for the height, one might want to ask whether there is any lower bound [2, 3, 4]. In fact, just looking at the lists of complete intersections in projective spaces (CICY [5, 6]) yields  $h^{11} + h^{21} \geq 30$  and the hypersurfaces in toric varieties satisfy  $h^{11} + h^{21} \geq 29$ . However, dividing out free group actions almost always lowers the Hodge numbers (and never raises them), so these naive lower bounds for the height can be easily violated [7, 8, 9]. From a physics perspective, this serves both to reduce the number of moduli [10, 11, 12, 13] and, via the Hosotani mechanism [14, 15, 16], to break the GUT gauge group. By systematically constructing free quotients of CICY threefolds [17, 3, 18, 19, 20, 21], one can push down the lower boundary for the height to  $h^{11} + h^{21} = 4$ . In particular, a minimal three-generation manifold [22] with  $(h^{11}, h^{21}) = (4, 1)$  can thus be realized.

However, one might wonder if even smaller Hodge numbers are possible. In particular, the minimal value for a non-rigid Calabi-Yau threefold would be  $(h^{11}, h^{21}) = (1, 1)$ . The purpose of this paper is to fill this gap, and construct a “minimal” Hodge number

example. The idea, in a nutshell, is to look for permutation actions that act simply transitively<sup>1</sup> on the vertices of a lattice polytope, and use this to define a group action on an anticanonical hypersurface in the corresponding toric variety. Those familiar with such constructions will immediately notice that this almost implies that there is a single complex structure modulus. However, various technical details need to be checked before one can conclude that this quotient is, indeed, a smooth Calabi-Yau threefold.

In Section 2, I will start with some elementary properties of the 24-cell lattice polytope that I will use in the following. In Section 3, I am going to define a group action and compute the fixed point sets on the ambient toric variety defined by the lattice polytope. Then, in Section 4, I will check that it leads to a desired free action on a Calabi-Yau hypersurface, leading to Hodge numbers  $(1, 1)$  on the smooth quotient threefold. Finally, in Section 5, I will quickly go through two similar group actions and all partial quotients. All toric geometry computations used in this paper were done using [23, 24, 25, 26].

## 2 The 24-Cell

There are six 4-dimensional regular polytopes, the 4-simplex, 4-cube, 16-cell, 24-cell, 120-cell and 600-cell. Apart from the 24-cell, these are higher-dimensional analogues of the tetrahedron, cube, octahedron, dodecahedron, and icosahedron. In particular, they transform in the same way under duality.<sup>2</sup> The 24-cell is the regular 4-dimensional polytope that does not have a 3-dimensional analog. For lack of anything else to transform into, it is also self-dual. A curious fact, that has already been remarked in [1], is that the 24-cell appears as one of the 473,800,776 reflexive 4-d lattice polytopes. In fact, the 24-cell can be constructed as the convex hull of the 24 roots of the  $D_4$  lattice. Amongst all 4-d reflexive lattice polytopes, it is the one with the largest symmetry group [1]. The symmetry group obviously must contain the Weyl group of  $D_4$ , but is actually larger. In fact, the full symmetry group of the 24-cell is the Weyl group of  $F_4$  and has 1152 elements.

## 3 A Toric Variety with $SL(2,3)$ Action

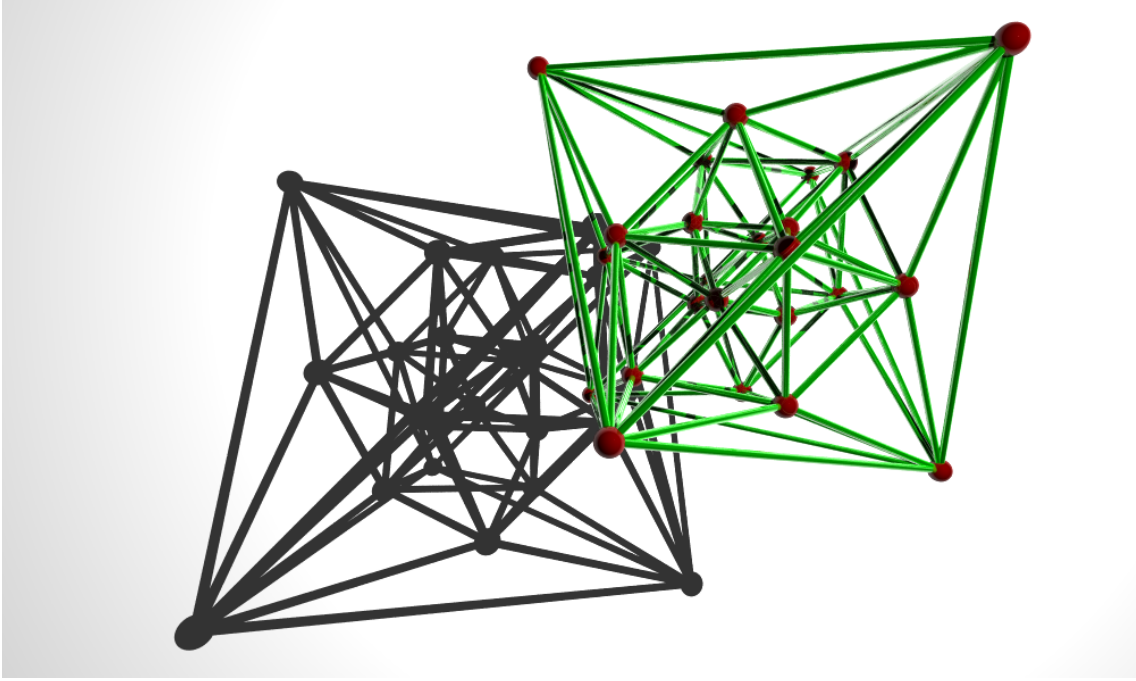
### 3.1 The Face Fan

Following the usual notation of toric geometry [27, 28, 29], we will identify the 4-dimensional root lattice of  $D_4$  with  $N \simeq \mathbb{Z}^4$ . Doing so breaks much of the symmetry

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<sup>1</sup>That is, for any two vertices  $v_1, v_2$  there exists a unique group element  $g \in G$  with  $g(v_1) = v_2$ .

<sup>2</sup>The tetrahedron is self-dual, cube and octahedron are exchanged, and dodecahedron and icosahedron are exchanged.



*Figure 2: Stereographic projection of the 24-cell into 3 dimensions.*

and the coordinates of the vertices do not manifest the 24-cell structure at all. However, picking a basis is convenient for direct computation and we will use the particular lattice basis of Tables 1 and 2 in the following. Given these 24 points, we define the polytope

$$\nabla = \text{conv} \{p_1, p_2, \dots, p_{24}\}. \quad (2)$$

Each of the 24 facets of  $\lambda$  is an octahedron, and spans one generating cone in the face fan

$$\begin{aligned} \mathcal{F}_\nabla = \{ & \langle p_2, p_6, p_{10}, p_{14}, p_{18}, p_{21} \rangle, \langle p_{10}, p_{12}, p_{14}, p_{16}, p_{21}, p_{24} \rangle, \langle p_2, p_{10}, p_{12}, p_{19}, p_{20}, p_{21} \rangle, \\ & \langle p_6, p_8, p_{14}, p_{16}, p_{21}, p_{23} \rangle, \langle p_2, p_4, p_6, p_8, p_{20}, p_{21} \rangle, \langle p_8, p_{12}, p_{16}, p_{20}, p_{21}, p_{22} \rangle, \langle p_1, p_5, p_9, p_{13}, p_{17}, p_{18} \rangle, \\ & \langle p_9, p_{11}, p_{13}, p_{15}, p_{17}, p_{24} \rangle, \langle p_1, p_3, p_9, p_{11}, p_{17}, p_{19} \rangle, \langle p_5, p_7, p_{13}, p_{15}, p_{17}, p_{23} \rangle, \langle p_1, p_3, p_4, p_5, p_7, p_{17} \rangle, \\ & \langle p_3, p_7, p_{11}, p_{15}, p_{17}, p_{22} \rangle, \langle p_9, p_{10}, p_{13}, p_{14}, p_{18}, p_{24} \rangle, \langle p_1, p_2, p_9, p_{10}, p_{18}, p_{19} \rangle, \langle p_9, p_{10}, p_{11}, p_{12}, p_{19}, p_{24} \rangle, \\ & \langle p_5, p_6, p_{13}, p_{14}, p_{18}, p_{23} \rangle, \langle p_1, p_2, p_4, p_5, p_6, p_{18} \rangle, \langle p_4, p_5, p_6, p_7, p_8, p_{23} \rangle, \langle p_{13}, p_{14}, p_{15}, p_{16}, p_{23}, p_{24} \rangle, \\ & \langle p_1, p_2, p_3, p_4, p_{19}, p_{20} \rangle, \langle p_{11}, p_{12}, p_{15}, p_{16}, p_{22}, p_{24} \rangle, \langle p_3, p_{11}, p_{12}, p_{19}, p_{20}, p_{22} \rangle, \langle p_7, p_8, p_{15}, p_{16}, p_{22}, p_{23} \rangle, \\ & \left. \langle p_3, p_4, p_7, p_8, p_{20}, p_{22} \rangle \right\}. \quad (3) \end{aligned}$$

We will now pick a particular 24-element subgroup of the automorphism group  $\text{Weyl}(F_4)$  of the 24-cell. Acting on from the left on the vertices  $p_i \in N$  of the polytope

$n$	$p_n$	$z_n$	$g_3(n)$	$g_4(n)$	facets
1	(1, 0, 0, 0)	$z_1$	14	10	STUVWX
2	(0, 1, 0, 0)	$z_2$	24	12	GORSTW
3	(0, 0, 1, 0)	$z_3$	18	9	EHNTUV
4	(0, 0, 0, 1)	$z_4$	10	19	GHPSTU
5	(1, -1, -1, 1)	$z_5$	21	2	DPQSUX
6	(0, 0, -1, 1)	$z_6$	12	20	GIPQRS
7	(0, -1, 0, 1)	$z_7$	2	1	DEHJPU
8	(-1, 0, 0, 1)	$z_8$	19	3	GHIJKP
9	(1, 0, 0, -1)	$z_9$	23	14	ABCVWX
10	(0, 1, 0, -1)	$z_{10}$	15	16	ABMORW
11	(0, 0, 1, -1)	$z_{11}$	5	13	BCELNV
12	(-1, 1, 1, -1)	$z_{12}$	17	15	BKLMNO
13	(1, -1, -1, 0)	$z_{13}$	8	6	ACDFQX
14	(0, 0, -1, 0)	$z_{14}$	22	8	AFIMQR
15	(0, -1, 0, 0)	$z_{15}$	4	5	CDEFJL
16	(-1, 0, 0, 0)	$z_{16}$	3	7	FIJKLM
17	(1, -1, 0, 0)	$z_{17}$	6	18	CDEUVX
18	(1, 0, -1, 0)	$z_{18}$	16	21	AQRSWX
19	(0, 1, 1, -1)	$z_{19}$	13	24	BNOTVW
20	(-1, 1, 1, 0)	$z_{20}$	9	11	GHKNOT
21	(-1, 1, 0, 0)	$z_{21}$	11	22	GIKMOR
22	(-1, 0, 1, 0)	$z_{22}$	1	17	EHJKLN
23	(0, -1, -1, 1)	$z_{23}$	20	4	DFIJPQ
24	(0, 0, 0, -1)	$z_{24}$	7	23	ABCFLM

**Table 1:** The vertices  $p_n$  of the 24-cell lattice polytope. The  $z_n$  column is the corresponding homogeneous variable of the toric variety, the next two columns are the transformation under the group action in eq. (5), and the final column lists the incident facets labeled as in Table 2.

$N$	equation
A	$(0, 0, -1, -1) \cdot \vec{x} = 1$
B	$(0, 0, 0, -1) \cdot \vec{x} = 1$
C	$(0, -1, 0, -1) \cdot \vec{x} = 1$
D	$(0, -1, 0, 0) \cdot \vec{x} = 1$
E	$(0, -1, 1, 0) \cdot \vec{x} = 1$
F	$(-1, -1, -1, -1) \cdot \vec{x} = 1$
G	$(0, 1, 0, 1) \cdot \vec{x} = 1$
H	$(0, 0, 1, 1) \cdot \vec{x} = 1$
I	$(-1, 0, -1, 0) \cdot \vec{x} = 1$
J	$(-1, -1, 0, 0) \cdot \vec{x} = 1$
K	$(-1, 0, 0, 0) \cdot \vec{x} = 1$
L	$(-1, -1, 0, -1) \cdot \vec{x} = 1$
M	$(-1, 0, -1, -1) \cdot \vec{x} = 1$
N	$(0, 0, 1, 0) \cdot \vec{x} = 1$
O	$(0, 1, 0, 0) \cdot \vec{x} = 1$
P	$(0, 0, 0, 1) \cdot \vec{x} = 1$
Q	$(0, 0, -1, 0) \cdot \vec{x} = 1$
R	$(0, 1, -1, 0) \cdot \vec{x} = 1$
S	$(1, 1, 0, 1) \cdot \vec{x} = 1$
T	$(1, 1, 1, 1) \cdot \vec{x} = 1$
U	$(1, 0, 1, 1) \cdot \vec{x} = 1$
V	$(1, 0, 1, 0) \cdot \vec{x} = 1$
W	$(1, 1, 0, 0) \cdot \vec{x} = 1$
X	$(1, 0, 0, 0) \cdot \vec{x} = 1$

**Table 2:** The facets of the 24-cell lattice polytope.

$\nabla$ , it is generated by the two matrices

$$g_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix}, \quad g_4 = \begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ -1 & -1 & -1 & -1 \end{pmatrix}. \quad (4)$$

Alternatively, the two generators can be written as the two permutations

$$\begin{aligned} g_3 &= (1, 14, 22)(2, 24, 7)(3, 18, 16)(4, 10, 15)(5, 21, 11)(6, 12, 17)(8, 19, 13)(9, 23, 20) \\ g_4 &= (1, 10, 16, 7)(2, 12, 15, 5)(3, 9, 14, 8)(4, 19, 24, 23)(6, 20, 11, 13)(17, 18, 21, 22) \end{aligned} \quad (5)$$

permuting the 24 vertices<sup>3</sup>. Together,  $g_3$  and  $g_4$  generate a representation of the group

$$G \stackrel{\text{def}}{=} \langle g_3, g_4 \rangle \simeq SL(2, 3), \quad (6)$$

of  $2 \times 2$ -matrices with entries in the finite field with 3 elements. Using the permutation group action, we can write the complete fan (including all lower-dimensional faces) succinctly as

$$\begin{aligned} \mathcal{F}_\nabla &= G \cdot \{ \langle p_1, p_2, p_3, p_4, p_{19}, p_{20} \rangle \} \cup \\ &\quad G \cdot \{ \langle p_1, p_2, p_4 \rangle, \langle p_1, p_3, p_4 \rangle, \langle p_1, p_2, p_{19} \rangle, \langle p_1, p_3, p_{19} \rangle \} \cup \\ &\quad G \cdot \{ \langle p_1, p_2 \rangle, \langle p_1, p_3 \rangle, \langle p_1, p_4 \rangle, \langle p_1, p_{19} \rangle \} \cup \\ &\quad G \cdot \{ \langle p_1 \rangle \} \cup \{ \langle \rangle \} \end{aligned} \quad (7)$$

See also Figure 3 for a pictorial representation of the relative position of the rays of the cones.

### 3.2 Homogeneous Coordinates and the Maximal Torus

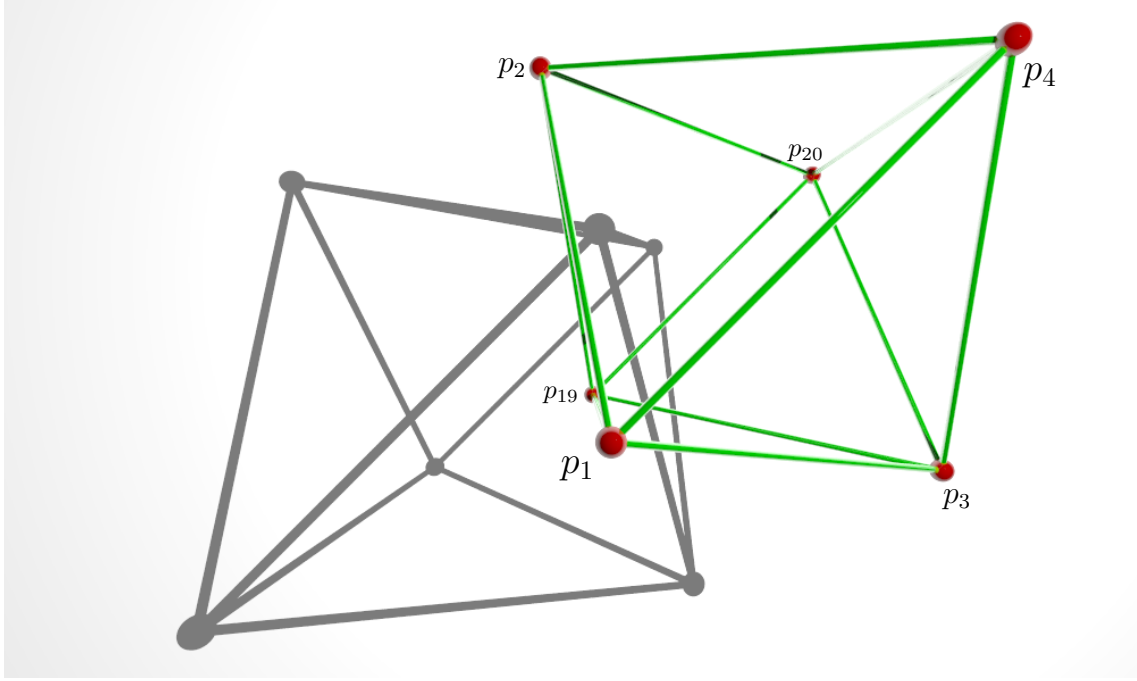
In the following, I will be using the Cox homogeneous coordinate description of the toric variety [30]. There are 24 homogeneous coordinates  $z_1, \dots, z_{24}$  modulo

$$\text{Hom}(A_3(\mathbb{P}_\nabla), \mathbb{C}^\times) = (\mathbb{C}^\times)^{20} \quad (8)$$

rescalings. By definition, the 4-dimensional toric variety  $\mathbb{P}_\nabla$  comes with an action of  $(\mathbb{C}^\times)^4$  such that there is a single maximal-dimensional orbit, which I will denote as  $\mathbb{P}_\emptyset$  in the following. In the usual correspondence between torus orbits and cones of the fan, this is the orbit associated to the trivial cone  $\langle \rangle$ . In terms of homogeneous coordinates, it is the locus where no homogeneous coordinate vanishes. The maximal orbit itself is always smooth as singular points must fill out whole torus orbits.

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<sup>3</sup>By abuse of notation, we will not distinguish the group from its matrix and permutation representation in the following.



**Figure 3:** One facet of the 24-cell with its spanning vertices.

In the remainder of this section, we will be investigating the fixed points of  $G \simeq SL(2, 3)$  on the maximal torus. Since all homogeneous coordinates are invertible there, we can freely use the homogeneous rescalings to set some homogeneous coordinates to unity. In fact, we can pick unique representatives

$$\begin{aligned} \mathbb{P}_{\langle \rangle} &= \left\{ [\hat{z}_1 : \cdots : \hat{z}_{24}] \mid \hat{z}_i \neq 0 \right\} = \\ &= \left\{ [z_1 : z_2 : z_3 : z_4 : 1 : \cdots : 1] \mid z_i \neq 0 \right\} \simeq (\mathbb{C}^\times)^4 \subset P_{\nabla} \end{aligned} \quad (9)$$

for each point where only the first four homogeneous coordinates are non-zero, and such that there are no remaining identifications. This is so because there is the following *integral basis* for the linear relations amongst the vertices of  $\nabla$ :

$$\begin{array}{ll} p_5 = & p_1 - p_2 - p_3 + p_4 & p_{15} = & -p_2 \\ p_6 = & & -p_3 + p_4 & p_{16} = & -p_1 \\ p_7 = & & -p_2 & +p_4 & p_{17} = & p_1 - p_2 \\ p_8 = & -p_1 & & +p_4 & p_{18} = & p_1 & -p_3 \\ p_9 = & p_1 & & -p_4 & p_{19} = & & p_2 + p_3 - p_4 \\ p_{10} = & & p_2 & -p_4 & p_{20} = & -p_1 + p_2 + p_3 \\ p_{11} = & & & p_3 - p_4 & p_{21} = & -p_1 + p_2 \\ p_{12} = & -p_1 + p_2 + p_3 - p_4 & p_{22} = & -p_1 & +p_3 \\ p_{13} = & p_1 - p_2 - p_3 & p_{23} = & & -p_2 - p_3 + p_4 \\ p_{14} = & & -p_3 & & p_{24} = & & -p_4 \end{array} \quad (10)$$



The first equation (for  $p_5$ ) then translates into the homogeneous rescaling

$$[z_1 : \cdots : z_{24}] = [\lambda z_1 : \lambda^{-1} z_2 : \lambda^{-1} z_3 : \lambda z_4 : \lambda^{-1} z_5 : z_6 : z_7 : \cdots : z_{24}] \quad (11)$$

and so on. Clearly, the basis for the relations eq. (10) allows us to unambiguously scale  $z_5, \dots, z_{24}$  to unity on the maximal torus.

### 3.3 Fixed Points on the Maximal Torus

To find the fixed points of the  $G$ -action on the toric variety  $\mathbb{P}_\nabla$ , we need to look at each conjugacy class of  $G$ . Excluding the identity of  $G$ , there are 6 non-trivial conjugacy classes. For the remainder of this section, I will discuss each in turn.

#### The conjugacy class of $g_4^2$

The first conjugacy class is of order 2 and contains the single central group element

$$g_4^2 = \text{diag}(-1, -1, -1, -1). \quad (12)$$

Its action on the homogeneous coordinates on the maximal torus  $\mathbb{P}_\diamond$  is

$$\begin{aligned} g_4^2([z_1 : z_2 : z_3 : z_4 : 1 : \cdots : 1]) &= \\ &= [1 : \cdots : 1 : \underbrace{z_3}_{\text{Position 14}} : \underbrace{z_2}_{\text{15}} : \underbrace{z_1}_{\text{16}} : 1 : \cdots : 1 : \underbrace{z_4}_{\text{Position 24}}] = \\ &= \left[ \frac{1}{z_1} : \frac{1}{z_2} : \frac{1}{z_3} : \frac{1}{z_4} : 1 : \cdots : 1 \right]. \end{aligned} \quad (13)$$

Hence, there are  $2^4$  fixed points

$$\mathbb{P}_\diamond^{g_4^2} = \left\{ [z_1 : z_2 : z_3 : z_4 : 1 : \cdots : 1] \mid z_0, z_2, z_2, z_3 \in \{+1, -1\} \right\} \quad (14)$$

#### The conjugacy class of $g_4$

The second conjugacy class is of order 4 and contains 6 group elements. Since a  $g_4$ -fixed point is also a  $g_4^2$ -fixed point, we immediately note that  $\mathbb{P}_\diamond^{g_4} \subset \mathbb{P}_\diamond^{g_4^2}$  is again a discrete set of points. Explicitly, the  $g_4$ -fixed point set turns out to be 4 out of the 12 fixed points of  $g_4^2$ , namely

$$\begin{aligned} \mathbb{P}_\diamond^{g_4} = \left\{ \begin{aligned} &[+1 : +1 : +1 : +1 : 1 : \cdots : 1], \\ &[+1 : -1 : -1 : +1 : 1 : \cdots : 1], \\ &[-1 : +1 : -1 : -1 : 1 : \cdots : 1], \\ &[-1 : -1 : +1 : -1 : 1 : \cdots : 1] \end{aligned} \right\}. \end{aligned} \quad (15)$$

### The conjugacy classes of $g_3$ and $g_3^2$

The third and fourth conjugacy class can be represented by  $g_3$  and  $g_3^2$ , respectively. They are both of order 3 and contain 4 representatives. In fact, they are related by an automorphism of  $G$  and, therefore, one only needs to discuss  $g_3$ , say. On the maximal torus  $\mathbb{P}_{\langle \rangle}$ , its action is

$$g_3\left([z_1 : z_2 : z_3 : z_4 : 1 : \cdots : 1]\right) = \left[z_3 : z_4 : \frac{1}{z_1 z_3} : \frac{1}{z_2 z_4} : 1 : \cdots : 1\right] \quad (16)$$

and, therefore, the fixed points are given by the ideal<sup>4</sup>

$$I^{g_3} = \left\langle z_1 = z_3, z_2 = z_4, z_3 = \frac{1}{z_1 z_3}, z_4 = \frac{1}{z_2 z_4} \right\rangle. \quad (17)$$

The associated variety  $V(I)$  is the fixed point set, and an elementary computation reveals that it consists of the 9 points

$$\mathbb{P}_{\langle \rangle}^{g_3} = V(I^{g_3}) = \left\{ [\mu : \nu : \mu : \nu : 1 : \cdots : 1] \mid \mu, \nu \in \{1, e^{2\pi i/3}, e^{4\pi i/3}\} \right\}. \quad (18)$$

### The conjugacy classes of $g_3 g_4^2$ and $g_3^2 g_4^2$

The fifth and sixth conjugacy class can be represented by  $g_3 g_4^2$  and  $g_3^2 g_4^2$ , respectively. They are both of order 6 and contain 4 representatives. These are again related by an automorphism of  $G$  and, therefore, one only needs to discuss

$$g_3 g_4^2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}. \quad (19)$$

Moreover,  $(g_3 g_4^2)^2 = g_3^2$ , so the fixed point set of  $g_3 g_4^2$  is contained in the 9 fixed points of  $g_3^2$ . An explicit computation shows that  $g_3 g_4^2$  fixes the single point

$$\mathbb{P}_{\langle \rangle}^{g_3 g_4^2} = \left\{ [1 : 1 : 1 : 1 : 1 : \cdots : 1] \right\} \quad (20)$$

on the maximal torus.

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<sup>4</sup>I will always think of the equations of the fixed points on the maximal torus as an ideal in the polynomial ring in the formal commutative variables  $z_1, z_1^{-1}, \dots, z_4, z_4^{-1}$  subject to the relations  $z_1 z_1^{-1} = 1, \dots, z_4 z_4^{-1} = 1$ .

### 3.4 Stanley-Reisner Ideal and Other Fixed Points

Thus far, I have shown that any element of  $G \simeq SL(2, 3)$  fixes a discrete set of points on the maximal torus  $\mathbb{P}_\diamond \simeq (\mathbb{C}^\times)^4$ , that is, on the locus where all homogeneous variables are non-zero. The 4-dimensional toric variety  $\mathbb{P}_\nabla$  is a compactification of  $\mathbb{P}_\diamond$  by gluing in lower-dimensional toric varieties associated to the non-empty cones of the fan  $\mathcal{F}_\nabla$ . One needs to discuss fixed points on these lower-dimensional strata as well. The main observation is that, given any  $g \in G$ , if the  $i$ -th homogeneous variable  $z_i$  vanishes, the permuted homogeneous variable  $z_{g(i)}$  has to vanish as well on the  $g$ -fixed point set. In other words, the  $g$ -fixed point sets on the lower-dimensional strata are contained in the subvarieties where whole  $g$ -permutation orbits of homogeneous variables vanish. As in the previous section, I will discuss each conjugacy class separately. Note that because  $G$  acts regularly (simply transitively) on the 24 variables, all  $g \in G$ -orbits are of the same size.

#### The conjugacy class of $g_4^2$

The 24 homogeneous variables  $z_1, \dots, z_{24}$  form 12 orbits of length 2 under the permutation  $g_4^2$ . For example, the orbit containing the first variable is  $\{z_1, z_{g_4^2(i)}\} = \{z_1, z_{16}\}$ . Therefore, the fixed point set away from the maximal torus is contained in the union of the 12 subvarieties

$$\left(\mathbb{P}_\nabla - \mathbb{P}_\diamond\right)^{g_4^2} \subset \bigcup_{g \in G} V(\langle z_{g(1)} = 0, z_{g(16)} = 0 \rangle). \quad (21)$$

#### The conjugacy class of $g_4$

The homogeneous variables form 6 orbits of length 4. The fixed point set away from the maximal torus is

$$\left(\mathbb{P}_\nabla - \mathbb{P}_\diamond\right)^{g_4} \subset \bigcup_{g \in G} V(\langle z_{g(1)} = 0, z_{g(7)} = 0, z_{g(10)} = 0, z_{g(16)} = 0 \rangle). \quad (22)$$

#### The conjugacy classes of $g_3$ and $g_3^2$

The homogeneous variables form 8 orbits of length 3. Again, it suffices to consider  $g_3$ -fixed points. Away from the maximal torus, they are

$$\left(\mathbb{P}_\nabla - \mathbb{P}_\diamond\right)^{g_3} \subset \bigcup_{g \in G} V(\langle z_{g(1)} = 0, z_{g(14)} = 0, z_{g(22)} = 0 \rangle). \quad (23)$$

## The conjugacy classes of $g_3g_4^2$ and $g_3^2g_4^2$

The homogeneous variables form 4 orbits of length 6. Again, it suffices to consider  $g_3g_4^2$ -fixed points. Away from the maximal torus, they are

$$\left(\mathbb{P}_\nabla - \mathbb{P}_\diamond\right)^{g_3g_4^2} \subset \bigcup_{g \in G} V(\langle z_{g(1)} = 0, z_{g(3)} = 0, z_{g(14)} = 0, \\ z_{g(16)} = 0, z_{g(18)} = 0, z_{g(22)} = 0 \rangle). \quad (24)$$

## The Stanley-Reisner Ideal

Not all homogeneous variables are allowed to vanish simultaneously as one can see from the homogeneous coordinate description

$$\mathbb{P}_\nabla = \frac{\mathbb{C}^{\mathcal{F}_\nabla(1)} - Z(\mathcal{F}_\nabla)}{(\mathbb{C}^\times)^{20}} \quad (25)$$

of the toric variety. Here,  $\mathcal{F}_\nabla(1)$  are the 24 rays of the fan,  $\mathbb{C}^{\mathcal{F}_\nabla(1)} \simeq \mathbb{C}^{24}$  is the affine space parametrized by the corresponding homogeneous variables, and  $Z(\mathcal{F}_\nabla)$  is the exceptional set

$$Z(\mathcal{F}_\nabla) = \bigcup_{z_{i_1} \cdots z_{i_k} \in SR(\mathcal{F}_\nabla)} V(z_{i_1} = 0, \dots, z_{i_k} = 0). \quad (26)$$

Taking the union over all monomials in the Stanley-Reisner ideal  $SR(\mathcal{F}_\nabla)$  is equivalent to only taking the union over the finitely many minimal monomials<sup>5</sup>

The Stanley-Reisner ideal of the variety  $\mathbb{P}_\nabla$  is generated by 204 monomials. Using the permutation group action, one can write it as

$$SR(\mathcal{F}_\nabla) = \left\langle \left\{ z_{g(1)}z_{g(16)}, z_{g(1)}z_{g(15)}, z_{g(1)}z_{g(14)}, z_{g(1)}z_{g(12)}, z_{g(1)}z_{g(24)}, \right. \right. \\ \left. \left. z_{g(1)}z_{g(6)}z_{g(20)}, z_{g(1)}z_{g(6)}z_{g(10)}, z_{g(1)}z_{g(6)}z_{g(7)}, z_{g(1)}z_{g(6)}z_{g(13)} \mid g \in G \right\} \right\rangle. \quad (27)$$

We notice that any potential fixed point in  $\mathbb{P}_\nabla - \mathbb{P}_\diamond$ , that is, outside of the maximal torus, is contained in the exceptional set, see eqns. (21), (22), (23), and (24). To summarize, all fixed point sets of all non-trivial group elements  $g \in G$  are finite<sup>6</sup> and contained in the maximal torus  $\mathbb{P}_\diamond$ ,

$$\bigcup_{g \in G - \{1\}} \mathbb{P}_\nabla^g \subset \mathbb{P}_\diamond \simeq (\mathbb{C}^\times)^4 \subsetneq \mathbb{P}_\nabla \quad (28)$$

<sup>5</sup>The “minimal” monomials are  $z_{i_1} \cdots z_{i_k}$  such that  $\{i_1, \dots, i_k\}$  is a primitive collection.

<sup>6</sup>Note that the fixed point set is automatically finite if it is contained in the maximal torus.

### 3.5 Singularities

Each of the 24 generating cones of the fan, for example  $\langle p_1, p_2, p_3, p_4, p_{19}, p_{20} \rangle$ , is not smooth.<sup>7</sup> This gives rise to 24 singular points of the toric variety  $\mathbb{P}_\nabla$ . All other cones of the fan (of dimension  $\leq 3$ ) are smooth, and, therefore,  $\mathbb{P}_\nabla$  is smooth outside of the 24 singular points.

In terms of homogeneous coordinates, these singular points are

$$\text{Sing}(\mathbb{P}_\nabla) = \left\{ V(z_{g(1)}, z_{g(2)}, z_{g(3)}, z_{g(4)}, z_{g(19)}, z_{g(20)}) \mid g \in G \right\}. \quad (29)$$

Since the maximal cones are not simplicial, the singular points are worse than orbifold singularities by a finite group.

In the remainder of this section I will investigate the singularities further. However, the details will not be important for the following. Now, since each singularity is purely local data, it is most convenient to use the description of the toric variety as patched local affine schemes  $\text{Spec}(\mathbb{C}[\sigma^\vee \cap M])$  instead of the global description via homogeneous coordinates. By symmetry, I just have to consider one of the 24 affine patches, and will pick

$$\begin{aligned} \sigma &= \langle p_1, p_2, p_3, p_4, p_{19}, p_{20} \rangle \\ \Leftrightarrow \sigma^\vee &= \langle (1, 1, 0, 1), (1, 1, 0, 0), (1, 0, 1, 0), (1, 0, 1, 1), \\ &\quad (0, 0, 1, 1), (0, 1, 0, 1), (0, 1, 0, 0), (0, 0, 1, 0) \rangle \end{aligned} \quad (30)$$

Since the dual cone  $\sigma^\vee$  is spanned by 8 rays, we need an 8-dimensional ambient affine space to embed the patch  $\text{Spec}(\mathbb{C}[\sigma^\vee \cap M])$  of  $\mathbb{P}_\nabla$ . A standard computation [31, 23] yields the toric ideal

$$\begin{aligned} \mathbb{C}[\sigma^\vee \cap M] &= \mathbb{C}[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8] / \\ &\quad \langle x_1x_3 - x_2x_4, x_1x_5 - x_4x_6, x_1x_7 - x_2x_6, x_1x_8 - x_2x_5, x_2x_5 - x_3x_6, \\ &\quad x_2x_5 - x_4x_7, x_2x_8 - x_3x_7, x_3x_5 - x_4x_8, x_5x_7 - x_6x_8 \rangle \end{aligned} \quad (31)$$

Note that the 9 defining equations are far from transverse and, in fact, cut out a 4-dimensional affine algebraic variety in  $\mathbb{C}^8$ . The singularity is at the origin  $x_1 = \dots = x_8 = 0$ .

## 4 The Calabi-Yau Threefold

### 4.1 Construction and Smoothness

We now pick a  $G \simeq SL(2, 3)$ -invariant section of the anti-canonical bundle on toric variety  $\mathbb{P}_\nabla$ , yielding a 3-dimensional variety  $\tilde{X}$  with vanishing first Chern class  $c_1(\tilde{X}) =$

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<sup>7</sup>A cone  $\sigma$  is smooth if the rays of the cone are a lattice basis for  $N \cap \text{span}_\mathbb{Q}(\sigma)$ . In other words, the associated open torus orbit is a smooth subset of the toric variety.

0. The sections of the anti-canonical bundle  $-K_{\nabla}$  of  $\mathbb{P}_{\nabla}$  are generated by the homogeneous monomials that correspond to the integral points of the dual polytope  $\Delta = \nabla^{\vee}$ , which are the origin and the 24 vertices of  $\Delta$  corresponding to the 24 facets of  $\nabla$ . That is [23],

$$H^0(\mathbb{P}_{\nabla}, -K_{\nabla}) = \left\langle \left\{ \prod_{i=1}^{24} z_i \right\} \cup \left\{ z_{g(1)}^2 z_{g(2)}^2 z_{g(3)}^2 z_{g(4)}^2 z_{g(5)} z_{g(6)} z_{g(7)} z_{g(8)} z_{g(9)} \times \right. \right. \\ \left. \left. z_{g(10)} z_{g(11)} z_{g(12)} z_{g(17)} z_{g(18)} z_{g(19)}^2 z_{g(20)}^2 z_{g(21)} z_{g(22)} \mid g \in G \right\} \right\rangle \quad (32)$$

The permutation action of  $G$  is facet-transitive on  $\nabla$  and, therefore, vertex-transitive on  $\Delta$ . Hence, the two invariant polynomials are the monomial corresponding to the origin,

$$P_{\infty} \stackrel{\text{def}}{=} \prod_{i=1}^{24} z_i, \quad (33)$$

and the sum over the 24 vertices of  $\Delta$ ,

$$P_0 \stackrel{\text{def}}{=} \sum_{g \in G} \left( z_{g(1)}^2 z_{g(2)}^2 z_{g(3)}^2 z_{g(4)}^2 z_{g(5)} z_{g(6)} z_{g(7)} z_{g(8)} z_{g(9)} \times \right. \\ \left. z_{g(10)} z_{g(11)} z_{g(12)} z_{g(17)} z_{g(18)} z_{g(19)}^2 z_{g(20)}^2 z_{g(21)} z_{g(22)} \right) \quad (34)$$

Together, they span the invariant sections

$$H^0(\mathbb{P}_{\nabla}, -K_{\nabla})^G = \text{span}_{\mathbb{C}}(P_0, P_{\infty}). \quad (35)$$

Hence, there is a one-parameter family

$$P_{\varphi} \stackrel{\text{def}}{=} P_0 + \varphi P_{\infty}, \quad \varphi \in \mathbb{C} \cup \{\infty\} \quad (36)$$

of invariant polynomials, giving rise to a family

$$\tilde{X}_{\varphi} \stackrel{\text{def}}{=} \{P_{\varphi} = 0\} \subset \mathbb{P}_{\nabla} \quad (37)$$

of  $G \simeq SL(2, 3)$ -symmetric varieties. The quotient  $X_{\varphi} = \tilde{X}_{\varphi}/G$  is then again a 3-dimensional variety with vanishing first Chern class  $c_1(X) = 0$ . There are 3 potential sources for singularities on the quotient, namely

1. singularities of the ambient toric variety  $\mathbb{P}_{\nabla}$  containing  $\tilde{X}$ ,
2. fixed points of the  $G$ -action on  $\tilde{X}$ , and
3. loci where the hypersurface equation  $P_{\varphi}$  fails to be transverse.

I will now discuss each case in turn.

1. Since  $G$  permutes the 24 singularities of  $\mathbb{P}_\nabla$ , one only has to consider one of them. For example, the fixed point corresponding to the cone  $\langle p_1, p_2, p_3, p_4, p_{19}, p_{20} \rangle \in \mathcal{F}_\nabla$  is

$$s = [0 : 0 : 0 : 0 : 1 : \cdots : 1 : \underbrace{0 : 0}_{\text{Positions 19 and 20}} : 1 : \cdots : 1] \in \text{Sing}(\mathbb{P}_\nabla), \quad (38)$$

see eq. (29). At the singular point,

$$P_0(s) = 1, \quad P_\infty(s) = 0. \quad (39)$$

Therefore, a sufficiently generic hypersurface  $\tilde{X}_\varphi$  misses the singular point  $s$ .

2. On the maximal torus  $\mathbb{P}_\diamond$  none of the homogeneous coordinates is allowed to vanish. In particular,

$$P_\infty(z) \neq 0 \quad \forall z \in \mathbb{P}_\diamond \quad (40)$$

by eq. (33). Since all fixed points are contained in  $\mathbb{P}_\diamond$ , a sufficiently generic hypersurface  $\tilde{X}_\varphi$  misses the fixed points. For example, one can check that  $\tilde{X}_1$  misses all fixed points.

3. Finally, one has to check that the hypersurface equation is transverse. As in the end of Subsection 3.5, I will make use of the covering of the toric variety  $\mathbb{P}_\nabla$  by 24 affine patches  $\text{Spec}(\mathbb{C}[\sigma^\vee \cap M])$  corresponding to the generating cones. By symmetry, one only has to check transversality in one of the 24 affine patches. In the patch used for eq. (31), the two invariant polynomials read

$$\begin{aligned} P_0|_{\text{Spec} \mathbb{C}[\sigma^\vee \cap M]} &= 1 + \sum_{i=1}^8 x_i + \\ &\quad x_1 x_3 + x_1 x_5 + x_2 x_6 + x_3 x_7 + x_3 x_5 + x_6 x_8 + x_1 x_3 x_7 + \\ &\quad x_3 x_6 (x_1 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_3 x_6) \\ P_\infty|_{\text{Spec} \mathbb{C}[\sigma^\vee \cap M]} &= x_3 x_6. \end{aligned} \quad (41)$$

A straightforward computation [25] yields that  $P_1 = P_0 + P_\infty$  is transverse. Therefore, any sufficiently generic  $P_\varphi$  is transverse.

This proves that a generic hypersurface  $X_\varphi$  is smooth. For example,  $X_1$  is smooth.

The (non-generic) hypersurfaces  $X_0$  and  $X_\infty$  are both singular because the special polynomials  $P_0$  and  $P_\infty$  fail to be transverse. Moreover,  $X_0$  has additional orbifold singularities because  $\tilde{X}_0$  passes through 12 out of the 16  $g_4^2$ -fixed points.<sup>8</sup> Finally,  $X_\infty$  has an additional singularity because  $\tilde{X}_\infty$  passes through the singular points of the ambient toric variety. A complete description of the complex structure moduli space will be given elsewhere [32].

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<sup>8</sup>The 4 points that are fixed by  $g_4^2$  and missed by  $\tilde{X}_0$  are the 4 fixed points of  $g_4$ .

## 4.2 Hodge Numbers

A generic (not necessarily symmetric) Calabi-Yau hypersurface in the toric variety  $\mathbb{P}_\nabla$  has Hodge numbers

$$h^{pq}(\tilde{X}) = \begin{array}{cccc} & & 1 & \\ & 0 & & 0 \\ & 0 & 20 & 0 \\ h^{pq}(\tilde{X}) = & 1 & 20 & 20 & 1 \\ & 0 & 20 & 0 \\ & 0 & & 0 \\ & & 1 & \end{array} . \quad (42)$$

Unsurprisingly, it is self-mirror ( $h^{11} = h^{21}$ ) since the 24-cell is a self-dual polytope. All complex structure deformations of  $\tilde{X}$  are represented by deformations of the defining polynomial. Therefore, the complex structure deformations of  $G$ -symmetric threefolds  $\tilde{X}$  are necessarily parametrized by the  $G$ -invariant polynomials. As we have seen in eq. (35), there is a one-dimensional family  $P_\varphi$  of invariant polynomials. Therefore,

$$h^{21}(X) = h^{21}(\tilde{X})^G = 1 \quad (43)$$

The Euler number  $0 = \chi(\tilde{X}) = \chi(X) = 2h^{11}(X) - 2h^{21}(X)$  then implies that  $h^{11}(X) = 1$  as well. To summarize, the Hodge diamond of a smooth quotient  $X = \tilde{X}/G$  Calabi-Yau threefold is

$$h^{pq}(X) = \begin{array}{cccc} & & 1 & \\ & 0 & & 0 \\ & 0 & 1 & 0 \\ h^{pq}(X) = & 1 & 1 & 1 & 1 \\ & 0 & 1 & 0 \\ & 0 & & 0 \\ & & 1 & \end{array} . \quad (44)$$

## 5 Generalizations

### 5.1 Permutation Orbifolds

Let me start by explaining some of the motivation behind the group action on the toric variety  $\mathbb{P}_\nabla$  defined by the 24-cell. By rewriting the Dolbeault cohomology and contracting with the covariant constant  $(3, 0)$ -form, the  $h^{21}$  complex structure moduli of the Calabi-Yau hypersurface  $\tilde{X}$  correspond to the tangent bundle-valued cohomology group

$$H^{2,1}(\tilde{X}) = H^1(\tilde{X}, \wedge^2 T^* \tilde{X}) = H^1(\tilde{X}, T\tilde{X}). \quad (45)$$

Now, the tangent bundle is a subbundle of the tangent bundle of the ambient space restricted to  $\tilde{X}$ ,

$$0 \longrightarrow T\tilde{X} \longrightarrow T\mathbb{P}_\nabla|_{\tilde{X}} \longrightarrow \mathcal{O}(-K_\nabla)|_{\tilde{X}} \longrightarrow 0. \quad (46)$$



This leads to the long exact sequence

$$\cdots \longrightarrow H^0(\tilde{X}, \mathcal{O}(-K_{\nabla})|_{\tilde{X}}) \longrightarrow H^1(\tilde{X}, T\tilde{X}) \longrightarrow H^1(\tilde{X}, T\mathbb{P}_{\nabla}|_{\tilde{X}}) = 0 \quad (47)$$

of cohomology groups, where I used the result for the tangent-bundle valued cohomology of the toric variety from Appendix A. We see that, *to minimize the surviving complex structure moduli on a free quotient  $X = \tilde{X}/G$ , one needs to minimize the number of  $G$ -invariant sections of the anticanonical bundle.*

The invariant sections of the (pull-back of the) anticanonical bundle are computed by the cohomology long exact sequence

$$0 \longrightarrow \underbrace{H^0(\mathbb{P}_{\nabla}, \mathcal{O})}_{\simeq \mathbb{C}} \longrightarrow \underbrace{H^0(\mathbb{P}_{\nabla}, \mathcal{O}(-K_{\nabla}))}_{\simeq \mathbb{C}^{\Delta \cap M}} \longrightarrow H^0(\tilde{X}, \mathcal{O}(-K_{\nabla})|_{\tilde{X}}) \longrightarrow 0. \quad (48)$$

The simplest group actions on toric varieties are toric actions, that is, subgroups of the maximal torus. But for these to admit a fixed-point free action on a Calabi-Yau hypersurface is very rare [33]. In particular, there is no such action on  $\mathbb{P}_{\nabla}$ . Given that the toric group actions do not suffice, one is led to consider permutation actions on the homogeneous coordinates coming from symmetries of the polytope. Note that the group elements, represented by orthogonal matrices, act in the same way on the dual polytopes  $\nabla \in N$  and  $\Delta \in M$  in order to preserve the inner product. Since we can identify

$$H^0(\mathbb{P}_{\nabla}, \mathcal{O}(-K_{\nabla})) = \mathbb{C}^{\Delta \cap M} \quad (49)$$

with the integral points of the polytope  $\Delta$ , there are always at least two invariant sections of the anticanonical bundle: The section corresponding to the origin of  $M$ , and the sum over one orbit of a vertex of  $\Delta$ . If the vertices form more than one orbit, then there are more invariant sections. Hence, one is led to search for subgroups of  $\text{Aut}(\Delta) = \text{Weyl}(F_4)$  that act simply transitively,<sup>9</sup> that is, with a single orbit of length 24, on the vertices of  $\nabla$ . A quick computation [26] reveals that, up to conjugacy, there are 22 different subgroups of order 24. Of these, 4 act simply transitively on the vertices. The groups are 2 subgroups isomorphic to  $SL(2, 3)$  that are not conjugate to each other, a semidirect<sup>10</sup> product  $\mathbb{Z}_3 \rtimes \mathbb{Z}_8$  with GAP id [24, 1], and  $\mathbb{Z}_3 \times Q_8$ .

Thus far, I only showed that the group action would lead to  $h^{2,1} = 1$  provided that the quotient  $X = \tilde{X}/G$  is smooth. But we are not guaranteed that it is smooth, and one must check it case-by-case. However, note that the invariant equations for all of these four group actions are the same, namely the one-parameter family  $P_{\varphi}$  defined in

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<sup>9</sup>It is, a priori, not impossible for groups with  $|G| > 24$  to act freely. The group action can then still be transitive but not simply transitive. Therefore, there is some  $g \in G$  that fixes some index  $i$ . The corresponding divisor  $z_i = 0$  is then mapped to itself by  $g$ , yet not forbidden by the Stanley-Reisner ideal. Hence, we cannot as easily conclude that all fixed point sets are discrete.

<sup>10</sup>To fix notation, I will say that  $G$  is a semidirect product of  $N$  and  $H$ , written  $N \rtimes H$ , if there is a group homomorphism  $G \rightarrow H$  with kernel  $N$ . For example,  $SL(2, 3)$  can be written as a semidirect product  $SL(2, 3) = Q_8 \rtimes \mathbb{Z}_3$ .

eq. (36). Hence, the covering Calabi-Yau manifold is always the same one-parameter family  $\tilde{X}_\varphi$ . We have already shown in Subsection 4.1 that a generic hypersurface does not meet the ambient singularities and is transverse. Therefore, the covering space is generically smooth and one only has to check that the group action is free on  $\tilde{X}$ . That is, the fixed point set in  $\mathbb{P}_\nabla$  must not meet the hypersurface for all  $g \in G - \{1\}$ . Direct computation shows that the other  $SL(2, 3)$ -subgroup, that is the one we have not used in Section 3, has fixed points.<sup>11</sup> The remaining two groups have, like the first  $SL(2, 3)$ , only isolated fixed points in the maximal torus and no fixed points outside of the maximal torus. Hence, all three groups act without fixed points on a generic hypersurface  $\tilde{X}$ .

To summarize, there are really three different free quotients of a generic invariant hypersurface  $\tilde{X} \subset \mathbb{P}_\nabla$ . The three quotients  $X_i = \tilde{X}/G_i$  are Calabi-Yau manifolds with the same Hodge numbers  $h^{11} = h^{21} = 1$ , but different fundamental groups. To remind ourselves, the first one was

$$G_1 = G = \langle g_3, g_4 \rangle \simeq SL(2, 3), \quad (50)$$

see eq. (5) for the definition of the permutations  $g_3$  and  $g_4$ . The second group can be written as a semidirect product

$$G_2 = \langle g_3, g_8 \rangle \simeq \mathbb{Z}_3 \rtimes \mathbb{Z}_8 \quad (51)$$

with

$$g_8 \stackrel{\text{def}}{=} (1, 2, 20, 12, 16, 15, 13, 5)(3, 10, 8, 11, 14, 7, 9, 6)(4, 19, 21, 22, 24, 23, 17, 18). \quad (52)$$

Finally, the third group can be written as a direct product

$$G_3 = \langle g_3, i, j \rangle \simeq \mathbb{Z}_3 \times Q_8 \quad (53)$$

with

$$\begin{aligned} i &= (1, 6, 16, 11)(2, 8, 15, 9)(3, 5, 14, 12)(4, 23, 24, 19)(7, 13, 10, 20)(17, 18, 21, 22), \\ j &= (1, 7, 16, 10)(2, 3, 15, 14)(4, 22, 24, 18)(5, 8, 12, 9)(6, 20, 11, 13)(17, 23, 21, 19). \end{aligned} \quad (54)$$

The three different groups have a common subgroup

$$\mathbb{Z}_6 = \langle g_3, g_4^2 \rangle = G_1 \cap G_2 \cap G_3 \quad (55)$$

which is normal in  $G_2$  and  $G_3$ , but not in  $G_1$ .

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<sup>11</sup>In fact, the other  $SL(2, 3)$ -action fixes surfaces in  $\mathbb{P}_\nabla$  intersecting  $\tilde{X}$  in curves.

$h^{11}(X_H)$	$h^{21}(X_H)$	$H$
20	20	1
12	12	$\mathbb{Z}_2$
8	8	$\mathbb{Z}_3$
6	6	$\mathbb{Z}_4$
4	4	$\mathbb{Z}_6$
3	3	$\mathbb{Z}_8, Q_8$
2	2	$\mathbb{Z}_{12}$
1	1	$SL(2, 3), \mathbb{Z}_3 \rtimes \mathbb{Z}_8, \mathbb{Z}_3 \times Q_8$

**Table 3:** Fundamental groups  $\pi_1(X_H) = H$  and Hodge numbers of the various free quotients of the Calabi-Yau hypersurface  $\tilde{X}$  in the toric variety  $\mathbb{P}_\nabla$ .

## 5.2 Subgroups and Partial Quotients

The three freely acting groups  $G_1$ ,  $G_2$ , and  $G_3$  on  $\tilde{X}$  have a various subgroups  $H$ , each of which acts freely and gives rise to another Calabi-Yau threefold  $X_H = \tilde{X}/H$ . Up to conjugation, the subgroups are

$$\begin{aligned}
G_1 &\supset 1, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6, Q_8, SL(2, 3); \\
G_2 &\supset 1, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6, \mathbb{Z}_8, \mathbb{Z}_{12}, \mathbb{Z}_3 \rtimes \mathbb{Z}_8; \\
G_3 &\supset 1, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4^{(1)}, \mathbb{Z}_4^{(2)}, \mathbb{Z}_4^{(3)}, \mathbb{Z}_6, Q_8, \mathbb{Z}_{12}^{(1)}, \mathbb{Z}_{12}^{(2)}, \mathbb{Z}_{12}^{(3)}, \mathbb{Z}_3 \times Q_8.
\end{aligned} \tag{56}$$

Note that  $G_3$  has three different  $\mathbb{Z}_4$  and three different  $\mathbb{Z}_{12}$  subgroups that are not conjugate to each other. The corresponding free quotients of  $\tilde{X}$  are not related by a symmetry of the covering space, and might be different manifolds.

Having identified the different subgroups  $H \subset G_i$ ,  $i = 1, 2, 3$ , one would like to know the Hodge numbers of the partial quotient Calabi-Yau threefolds. Again because the Euler number vanishes, it suffices to compute

$$h^{21}(X_H) = h^{21}(\tilde{X}/H) = \dim H^1(\tilde{X})^H. \tag{57}$$

From the computation of the tangent bundle cohomology in Appendix A, one can identify

$$H^1(\tilde{X}) = \mathbb{C}^{\mathcal{F}_\nabla(1)} - N \otimes_{\mathbb{Z}} \mathbb{C} \tag{58}$$

as  $H$ -representations. That is,

- $\mathbb{C}^{\mathcal{F}_\nabla(1)}$  is the 24-dimensional representation spanned by the 24 vertices of the polytope  $\nabla$ , and
- $N \otimes_{\mathbb{Z}} \mathbb{C}$  is the 4-dimensional matrix representation of the group.

It is then an easy exercise to compute the invariant cohomology groups. It turns out that the representation on  $H^1(T\tilde{X})$  only depends on the group, and not on the details of the permutation action. The resulting Hodge numbers, for the different groups, are listed in Table 3.

## A Cohomology of the Tangent Bundle

### A.1 Of the Ambient Toric Variety

The tangent bundle of a toric variety has a monad presentation

$$0 \longrightarrow (n-d)\mathcal{O} \longrightarrow \bigoplus_{i=1}^n \mathcal{O}(V(z_i)) \longrightarrow T\mathbb{P}_\nabla \longrightarrow 0 \quad (59)$$

In our particular case, one only has to be careful about the singularities. For example,  $\mathcal{O}(V(z_i))$  is not a line bundle, but only a reflexive sheaf. Nevertheless, we can apply exact sequences. Using the standard toric algorithm for the cohomology of Weil divisors [29], I find

$$\begin{aligned} h^\bullet(\mathbb{P}_\nabla, \mathcal{O}) &= h^\bullet(\mathbb{P}_\nabla, \mathcal{O}(V(z_i))) = (1, 0, 0, 0, 0), \\ h^\bullet(\mathbb{P}_\nabla, \mathcal{O}(K_\nabla)) &= (0, 0, 0, 0, 1), \\ h^\bullet(\mathbb{P}_\nabla, \mathcal{O}(V(z_i)) \otimes \mathcal{O}(K_\nabla)) &= (0, 0, 0, 0, 0). \end{aligned} \quad (60)$$

Therefore,

$$\begin{aligned} h^\bullet(\mathbb{P}_\nabla, T\mathbb{P}_\nabla) &= (4, 0, 0, 0, 0), \\ h^\bullet(\mathbb{P}_\nabla, T\mathbb{P}_\nabla \otimes \mathcal{O}(K_\nabla)) &= (0, 0, 0, 20, 0). \end{aligned} \quad (61)$$

The restriction to a smooth anticanonical hypersurface  $\tilde{X}$  can be computed from the short exact sequence

$$0 \longrightarrow T\mathbb{P}_\nabla \otimes K_\nabla \longrightarrow T\mathbb{P}_\nabla \longrightarrow T\mathbb{P}_\nabla|_{\tilde{X}} \longrightarrow 0, \quad (62)$$

and I find

$$h^\bullet(\tilde{X}, T\mathbb{P}_\nabla|_{\tilde{X}}) = (4, 0, 20, 0). \quad (63)$$

### A.2 Of the Hypersurface

First, we need the cohomology of the restriction  $\mathcal{O}(-K_\nabla)|_{\tilde{X}}$  of the anticanonical bundle. Analogous to eq. (62), I find

$$h^0(\tilde{X}, \mathcal{O}(-K_\nabla)|_{\tilde{X}}) = h^0(\mathbb{P}_\nabla, \mathcal{O}(-K_\nabla)) - h^0(\mathbb{P}_\nabla, \mathcal{O}) = 25 - 1 = 24, \quad (64)$$

and all higher cohomology groups vanish. Therefore, the tangent bundle of the hypersurface,

$$0 \longrightarrow T\tilde{X} \longrightarrow T\mathbb{P}_{\nabla}|_{\tilde{X}} \longrightarrow \mathcal{O}(-K_{\nabla})|_{\tilde{X}} \longrightarrow 0 \quad (65)$$

has cohomology groups

$$h^{\bullet}(\tilde{X}, T\tilde{X}) = (0, 20, 20, 0) \quad (66)$$

## Bibliography

- [1] M. Kreuzer and H. Skarke, “Complete classification of reflexive polyhedra in four dimensions,” *Adv. Theor. Math. Phys.* **4** (2002) 1209–1230, [hep-th/0002240](#). 1, 2
- [2] P. Candelas, X. de la Ossa, Y.-H. He, and B. Szendroi, “Triadophilia: A Special Corner in the Landscape,” [arXiv:0706.3134](#) [[hep-th](#)]. 1
- [3] P. Candelas and R. Davies, “New Calabi-Yau Manifolds with Small Hodge Numbers,” *Fortsch. Phys.* **58** (2010) 383–466, [0809.4681](#). 1
- [4] P. Candelas and A. Constantin, “Completing the Web of  $Z_3$  - Quotients of Complete Intersection Calabi-Yau Manifolds,” [1010.1878](#). 1
- [5] P. Candelas, A. M. Dale, C. A. Lutken, and R. Schimmrigk, “Complete Intersection Calabi-Yau Manifolds,” *Nucl. Phys.* **B298** (1988) 493. 1
- [6] P. Candelas, C. A. Lutken, and R. Schimmrigk, “Complete Intersection Calabi-Yau Manifolds. 2. Three Generation Manifolds,” *Nucl. Phys.* **B306** (1988) 113. 1
- [7] R. Davies, “Quotients of the conifold in compact Calabi-Yau threefolds, and new topological transitions,” [0911.0708](#). 1
- [8] R. Davies, “Hyperconifold Transitions, Mirror Symmetry, and String Theory,” [1102.1428](#). \* Temporary entry \*. 1
- [9] S. Filippini and A. Garbagnati, “A rigid Calabi–Yau 3-fold,” [1102.1854](#). 1
- [10] V. Braun, B. A. Ovrut, T. Pantev, and R. Reinbacher, “Elliptic Calabi-Yau threefolds with  $Z(3) \times Z(3)$  Wilson lines,” *JHEP* **12** (2004) 062, [hep-th/0410055](#). 1
- [11] V. Braun, M. Kreuzer, B. A. Ovrut, and E. Scheidegger, “Worldsheet Instantons, Torsion Curves, and Non-Perturbative Superpotentials,” *Phys. Lett.* **B649** (2007) 334–341, [hep-th/0703134](#). 1

- [12] V. Braun, M. Kreuzer, B. A. Ovrut, and E. Scheidegger, “Worldsheet instantons and torsion curves. Part A: Direct computation,” *JHEP* **10** (2007) 022, [hep-th/0703182](#). 1
- [13] V. Braun, M. Kreuzer, B. A. Ovrut, and E. Scheidegger, “Worldsheet Instantons and Torsion Curves, Part B: Mirror Symmetry,” *JHEP* **10** (2007) 023, [arXiv:0704.0449 \[hep-th\]](#). 1
- [14] Y. Hosotani, “Dynamical Mass Generation by Compact Extra Dimensions,” *Phys. Lett.* **B126** (1983) 309. 1
- [15] Y. Hosotani, “Dynamics of Nonintegrable Phases and Gauge Symmetry Breaking,” *Ann. Phys.* **190** (1989) 233. 1
- [16] E. Witten, “Symmetry Breaking Patterns in Superstring Models,” *Nucl. Phys.* **B258** (1985) 75. 1
- [17] M. Gross and S. Pavanelli, “A Calabi-Yau threefold with Brauer group  $(\mathbb{Z}/8\mathbb{Z})^2$ ,” [math.AG/0512182](#). 1
- [18] M. Gross and S. Popescu, “Calabi-Yau Threefolds and Moduli of Abelian Surfaces I,” *ArXiv Mathematics e-prints* (Jan., 2000) [arXiv:math/0001089](#). 1
- [19] L. Borisov and Z. Hua, “On Calabi-Yau threefolds with large nonabelian fundamental groups,” *Proc. Amer. Math. Soc.* **136** (2008), no. 5, 1549–1551, [arXiv:math/0609728](#). 1
- [20] Z. Hua, “Classification of free actions on complete intersections of four quadrics,” [0707.4339](#). 1
- [21] V. Braun, “On Free Quotients of Complete Intersection Calabi-Yau Manifolds,” [1003.3235](#). 1
- [22] V. Braun, P. Candelas, and R. Davies, “A Three-Generation Calabi-Yau Manifold with Small Hodge Numbers,” *Fortsch. Phys.* **58** (2010) 467–502, [0910.5464](#). 1
- [23] V. Braun and A. Novoseltsev, “Toric Geometry in the Sage CAS.” to appear. 1, 3.5, 4.1
- [24] W. Stein *et al.*, *Sage Mathematics Software (Version 4.6.2)*. The Sage Development Team, 2011. <http://www.sagemath.org>. 1
- [25] G.-M. Greuel, G. Pfister, and H. Schönemann, “SINGULAR 3.0,” a computer algebra system for polynomial computations, Centre for Computer Algebra, University of Kaiserslautern, 2005. <http://www.singular.uni-kl.de>. 1, 3

- [26] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.4.12*, 2008. <http://www.gap-system.org>. 1, 5.1
- [27] W. Fulton, *Introduction to Toric Varieties*. Princeton University Press, 1993. 3.1
- [28] V. V. Batyrev, “Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties,” *J. Alg. Geom.* **3** (1994) 493–545. 3.1
- [29] D. A. Cox, J. B. Little, and H. K. Schenck, *Toric Varieties*. AMS, 2011. 3.1, A.1
- [30] D. A. Cox, “The homogeneous coordinate ring of a toric variety,” *J. Algebraic Geom.* **4** (1995), no. 1, 17–50. 3.2
- [31] S. Hosten and B. Sturmfels, “GRIN: An Implementation of Gröbner Bases for Integer Programming,” 1995. 3.5
- [32] “Mirror Symmetry of the Minimal Calabi-Yau Manifolds.” to appear. 4.1
- [33] V. Batyrev and M. Kreuzer, “Integral cohomology and mirror symmetry for Calabi-Yau 3-folds,” in *Mirror symmetry. V*, vol. 38 of *AMS/IP Stud. Adv. Math.*, pp. 255–270. Amer. Math. Soc., Providence, RI, 2006. 5.1